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G. A. Konstantinov and Yu. L. Yakimov

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NUMERICAL METHOD FOR SOLVING NON-STATIONARY AXISYMMETRIC PROBLEMS OF HYDRODYNAMICS OF AN IDEAL FLUID WITH FREE SURFACES

G. A. Konstantinov and Yu. L. Yakomov
(Moscow)

ABSTRACT. A method is proposed for the solution of problems concerning non-stationary axisymmetric potential flow of an ideal incompressible fluid with a free surface. The method is based on the discrete distribution of annular sources on boundary surfaces, which involves the reduction of the problem in question to a system of ordinary differential equations.

An approximation method is described for the solution of problems in non- /16
stationary axisymmetric potential flow of an ideal incompressible fluid with a free surface. The method is based on the discrete distribution of annular sources on boundary surfaces which permits the problem under consideration to be reduced to a system of ordinary differential equations. The basis of this numerical method is the known result, in agreement with which any harmonic function may be represented as the potential of sources distributed on a flow boundary [1,2]. This work contains a numerical solution of the problem in which the free surface will be the surface of a gas cavity.

1. The problem is considered of the location of the surface of a gas cavity which is formed at the end of a round pipe protruding from an infinite wall. The surface of the cavity is essentially non-stationary due to the pressure difference in the pipe and in the surrounding fluid. The fluid is assumed to be ideal, incompressible and quiescent at infinity. It is assumed that the z axis coincides with the axis of flow symmetry and with the direction of the force of gravity, and also that the infinite wall lies in the plane z = 0. The presence of an infinite wall is equivalent to flow symmetry with respect to this plane (Figure 1).

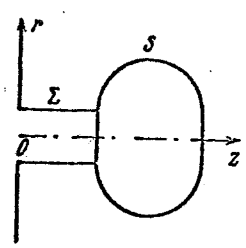

Figure 1.

The potential u of the fluid traveling speed will be a function of cylindrical coordinates r, z and time t, where r is the distance to the axis of symmetry. The function u(r, z, t) satisfies the Laplace equation in cylindrical coordinates for the axisymmetrical case

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0 \qquad (1.1)$$

*Numbers in the margin indicate pagination in the foreign text.

and with boundary conditions

$$\frac{\partial u}{\partial n}=0 \tag{1.2}$$

on the surface Σ of the pipe

$$\frac{\partial u}{\partial t}+\frac{p(t)-p_{\infty}}{\rho}+\frac{1}{2}v(t)^2-gz=0 \tag{1.3}$$

on free surface S. Here p(t) is pressure in the cavity; p_∞ is pressure at a point infinitely removed from the axis of symmetry and which lies in the plane z = 0; v(t) is the magnitude of the displacement velocity of surface S; g is the value of the acceleration of gravity; ρ is fluid density.

In order to obtain the differential motion equations of the free surface we employ the condition of potential flow /163

$$\mathbf{v}=\operatorname{grad} u$$

2. The boundary surfaces S and Σ with planes parallel to the plane z = 0 are laid out on the rings S_j (j = 0.1,...,N), which cover the surfaces S and Σ. From each surface S_j the circumference of radius $R_j(Z_j)$ is taken, the potential of which has linear density Q_j. The potential u(r,z,t) at the point (R_j, Z_j) of the meridian half-plane zr is sought in the form of the sum of the potentials, taken in this manner, of the circumferences with radii R_i, situated on rings S_i

$$u(r,z,t)\,|\,R_j,\ Z_j=-\sum_{i=0}^{N}R_i(t)Q_i(t)\,s_i(r,z,t)\,|\,R_j,\ Z_j \tag{2.1}$$

which is equivalent to an approximation of the surface potential of a simple layer, which is the solution of equation (1.1), and the final sum according to the rectangle method. Here

$$s_i(r,z,t)\,|\,R_j,\ Z_j=\int_0^{2\pi}\{[r^2+R_i^2-2rR_i\cos\alpha+(z-Z_i)^2]^{-1/2}+ [r^2+R_i^2-2rR_i\cos\alpha+(z+Z_i)^2]^{-1/2}\}\,d\alpha\,|\,R_j,\ Z_j \qquad (i\neq j) \tag{2.2}$$

$$s_j(r,z,t)\,|\,R_j,Z_j=2\pi\left(\frac{2\alpha_j}{\sigma_j}\right)^{1/2}-\frac{2}{R_j}\ln\left|\tan\frac{\alpha_j}{8}\right|+\int_0^{2\pi}[2R_j^2(1-\cos\alpha)+(2Z_j)^2]^{-1/2}d\alpha \tag{2.3}$$

in which equation (2.3) will be an approximation of the value of the potential of ring S_j with unity density. This approximation is made in the following manner: ring S_j, by means of central angle θ < α_j < 2π (Figure 2) is divided into two parts, and the potential of that part of the ring which is based on angle α_j is replaced by the potential of the circle with radius ρ_j with center at point (R_j, Z_j), tangent to the surface S_j at this point; the potential of the remaining part of the ring is replaced by the potential of the arc of the circumference with radius R_j with central angle $2\pi - \alpha_j$, which is located on the ring. The radius ρ_j of the circle is determined from requirements so that its area will be $2\pi/\alpha_j$ times less than the area σ_j of the ring, from which

$$\rho_j = (\alpha_j \sigma_j / 2\pi^2)^{1/2}.$$

Equation (2.3) is obtained by a limiting process from the equation for the potential of the system of the circle and the arc of the circumference chosen in this manner during the approach of arbitrary point (r,z) to point (R_j, Z_j), which is located on the surface S_j.

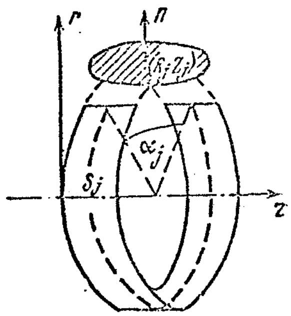

Figure 2.

Keeping in mind that an exact solution for ring potential does not depend on angle α_j, which is introduced only in order to obtain an approximate solution, we shall require that the derivative with respect to α_j from equation (2.3) equal 0. This leads to the equation

$$\sqrt{\alpha_j} - 2\pi R_j \sqrt{\frac{2_j}{\sigma_j}} \sin \frac{\alpha_j}{4} = 0 \qquad (2.4)$$

which has a single root $0 < \alpha_j < 2\pi$, if the following relationship is fulfilled

$$\sigma_j < 4\pi R_j^2 . \qquad (2.5)$$

Thus under conditions (2.3) the angle α_j is uniquely determined from equation (2.4) as a function of coordinates R_j and Z_j. In introducing the following designations:

$$q_i = Q_i R_i, \qquad a_i^2 = r^2 + R_i^2 + (z - Z_i)^2, \qquad a_i^{*2} = r^2 + R_i^2 + (z + Z_i)^2$$

$$k_i = \frac{2rR_i}{a_i^2}, \qquad k_i^* = \frac{2rR_i}{a_i^{*2}}, \qquad f_1(k) = \int_0^\pi (1 - k\cos\alpha)^{-1/2} d\alpha$$

we have /164

$$s_i(r, z, t) \mid R_j, Z_j = 2\left[\frac{f_1(k_i)}{a_i} + \frac{f_1(k_i^*)}{a_i^*}\right] \Bigg| R_j, Z_j \qquad (i \neq j)$$

$$s_j(r, z, t) \mid R_j, Z_j = 2\left[\pi\left(\frac{2\alpha_j}{\sigma_j}\right)^{1/2} - \frac{1}{R_j}\ln\left|\tan\frac{\alpha_j}{8}\right| + \frac{f_1(k_j^*)}{a_j^*}\right]$$

and

$$u(r, z, t) \mid R_j, Z_j = -\sum_{i=0}^{N} q_i(t)\, s_i(r, z, t) \mid R_j, Z_j \qquad (2.6)$$

$$(j = 0, 1, \ldots, N)$$

Similarly for the derivatives of potential u(r,z,t) we obtained the following equations according to the space variables r and z:

$$\left.\frac{\partial u}{\partial r}\right|_{Rj, Z_j} = -\sum_{i=0}^{N} q_i \left.\frac{\partial s_i}{\partial r}\right|_{R_j, Z_j} \qquad (2.7)$$

$$\left.\frac{\partial u}{\partial z}\right|_{R_j, Z_j} = -\sum_{i=0}^{N} q_i \left.\frac{\partial s_i}{\partial z}\right|_{R_j, Z_j} \qquad (2.8)$$

$$(j = 0, 1, \ldots, N)$$

where

$$\left.\frac{\partial s_i}{\partial r}\right|_{R_j, Z_j} = -2\left[\frac{rf_2(k_i) - R_i f_3(k_i)}{a_i^3} + \frac{rf_2(k_i^*) - R_i f_3(k_i^*)}{a_i^{*3}}\right]\Bigg|_{R_j, Z_j\,(i\neq j)}$$

$$\left.\frac{\partial s_j}{\partial r}\right|_{R_j, Z_j} = -4\pi^2 \frac{1}{\sigma_j}\cos\theta_j + \frac{1}{R_j^2}\ln\left|\tan\frac{\alpha_j}{8}\right| - 2\frac{R_i[f_2(k_i^*) - f_3(k_i^*)]}{a_i^{*3}}$$

$$\left.\frac{\partial s_i}{\partial z}\right|_{R_j, Z_j} = -2\left[\frac{(z - Z_i) f_2(k_i)}{a_i^3} + \frac{(z + Z_i) f_2(k_i^*)}{a_i^{*3}}\right]\Bigg|_{R_j, Z_j\,(i\neq j)}$$

$$\left.\frac{\partial s_j}{\partial z}\right|_{R_j, Z_j} = 4\pi^2 \frac{1}{\sigma_j}\sin\theta_j - \frac{4z_j f_2(k_j^*)}{a_j^{*3}}$$

$$f_2(k) = \int_0^{\pi} (1 - k\cos\alpha)^{-3/2}\, d\alpha, \qquad f_3(k) = \int_0^{\pi} \cos\alpha\,(1 - k\cos\alpha)^{-3/2}\, d\alpha$$

Here θ_j is the angle between the z axis and the tangent at the point (R_j,Z_j) to the meridian section of the surface S and Σ, located in the sectional plane.

The fulfillment of the boundary condition (1.2) on the surface Σ is equivalent to the fulfillment of condition

$$\sum_{i=0}^{N} q_i (s_{rij} c_{rj} + s_{zij} c_{zj}) = 0 \qquad (j = M + 1, \ldots, N) \qquad (2.9)$$

in points (R_j,Z_j), where c_{rj} = cos (n,r), c_{zj} = cos (n,z) are direction cosines of the external normal to the surface Σ at the point (R_j,Z_j); s_{rij}, s_{zij} are values of the functions $\partial s_i/\partial r$, $\partial s_i/\partial z$ at the same point. Designating the function u(r,z,t) at the point (R_j,Z_j) by $u_j(t)$, we have

$$u_j(t) = -\sum_{i=0}^{N} q_i(t)\, s_{ij}(t) \qquad (j = 0, 1, \ldots, M) \tag{2.10}$$

The values $q_i(t)$ $(i = 0, 1, \ldots, N)$ are determined from conditions (2.9), (2.10).

Considering the potential flow condition (1.4), the boundary condition (1.3) as well as (2.10), and passing to dimensionless values

$$\varphi = \frac{u}{v_0 L}, \quad \eta = \frac{r}{L}, \quad \zeta = \frac{z}{L}, \quad \tau = \frac{t v_0}{L}, \quad \mu = \frac{q}{v_0 L^2}$$

we have the following system of ordinary differential equations: /165

$$\begin{aligned}
\frac{d\eta_j(\tau)}{d\tau} &= -a_j(\tau) \sum_{i=0}^{N} \mu_i(\tau)\, s_{\eta ij}(\tau) \qquad (j = 0, 1, \ldots, M+1)\\
\frac{d\zeta_j(\tau)}{d\tau} &= -b_j(\tau) \sum_{i=0}^{N} \mu_i(\tau)\, s_{\zeta ij}(\tau) \qquad (j = 0, 1, \ldots, M+1)\\
\frac{d\varphi_j(\tau)}{d\tau} &= -\frac{\Delta p(\tau)}{\Delta p(0)} + \frac{1}{2}\left[\left(\frac{d\eta_j}{d\tau}\right)^2 + \left(\frac{d\zeta_j}{d\tau}\right)^2\right] + \lambda \zeta_j \qquad (j = 0, 1, \ldots, M)
\end{aligned} \tag{2.11}$$

Here

$$a_j(\tau) = \begin{cases} 1 & \text{where } j = 0, 1, \ldots, M \\ -c_{\zeta j} & \text{where } j = M+1 \end{cases} \qquad b_j(\tau) = \begin{cases} 1 & \text{where } j = 0, 1, \ldots, M \\ c_{\eta j} & \text{where } j = M+1 \end{cases}$$

$$s_{\eta ij} = \left.\frac{\partial s_i}{\partial \eta}\right|_{\eta_j, \zeta_j}, \qquad s_{\zeta ij} = \left.\frac{\partial s_i}{\partial \zeta}\right|_{\eta_j, \zeta_j}.$$

$$\Delta p(\tau) = p(\tau) - p_\infty, \qquad \lambda = \frac{2gL}{v_0^2}, \qquad v_0 = \left(\frac{\Delta p(0)}{\rho}\right)^{1/2}.$$

L is a characteristic linear dimension.

It is assumed that for $j = M + 2, \ldots, N$ ring sources on the surface Σ are distributed arbitrarily provided that condition (2.9) is fulfilled. In this work the values for quantities $\eta_j(\tau)$, $\zeta_j(\tau)$ for a case where the surface Σ is a pipe with length $l \neq 0$, are calculated from the relationship

$$\begin{aligned}
\eta_j &= \eta_{M+1} + 2^{j-(M+2)} h_1\\
\zeta_j &= \zeta_{M+1} + 2^{j-(M+2)} h_2 \qquad (j = M+1, \ldots, N)\\
h_1 &= \frac{\eta_N - \eta_{M+1}}{20l}, \qquad h_2 = \frac{\zeta_N - \zeta_{M+1}}{20l}
\end{aligned}$$

Integration of the system (2.11) permits us to make an approximate determination of surface $S(\tau)$, and therefore also of potential $\phi(\eta, \zeta, \tau)$.

3. The system (2.11) of ordinary differential equations was integrated by the numerical Runge-Kutta method on an M-20 EVM [electronic computer] with constant interval h with respect to dimensionless time τ, where it was assumed that $\phi_j(0) = 0$ $(j = 0, 1, \ldots, M)$ and $g = 0$. The initial values of the free surface coordinates were established on a pipe shear plane $z = l$. A case was examined for inertial expansion of a cavity under the influence of an initial surplus pressure momentum, where

$$\Delta p(\tau) = \begin{cases} \Delta p(0) & \text{where } \tau \leqslant h \\ 0 & \text{where } \tau > h \end{cases}$$

Here $h = 0.1$, $\Delta p(0) = 1.02 \cdot 10^4$, $M = 8$, $N = 17$, $c_{\zeta j} = 0$, $c_{\eta j} = 1$ $(j = M + 1, \ldots, N)$, $l = 2R$ (l, R are the length and radius of the pipe, respectively). Based on numerical calculations Figure 3 shows a representation of the change in cavity surface during the process of inertial expansion.

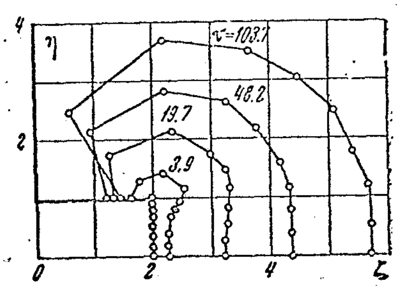

Figure 3.

In order to compare the numerical solution obtained with known exact solutions, a problem was examined concerning the expansion of a spherical gas cavity in an unbounded fluid. Cases were examined involving cavity expansion with constant surplus pressure, inertial expansion, and harmonic vibrations of the cavity with an adiabatic pressure change law. In all of these cases the maximum relative error in the calculation of free surface coordinates where $M = N = 6$, $h \geq 0.02$ did not exceed 3%.

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